Isomorphism Theorem

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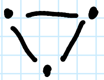
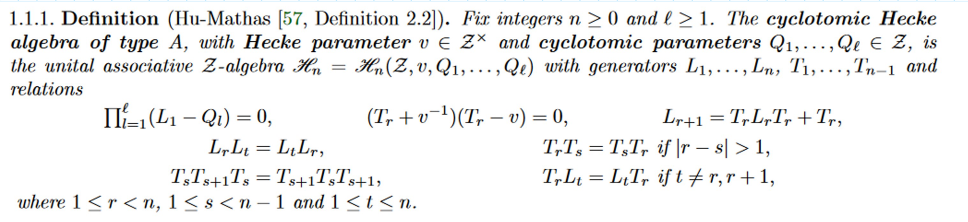
Reminder: $\Gamma_e =$  $= \hat{A}_{e-1}$, $I_e = \mathbb{Z}/e\mathbb{Z} =$ vertices



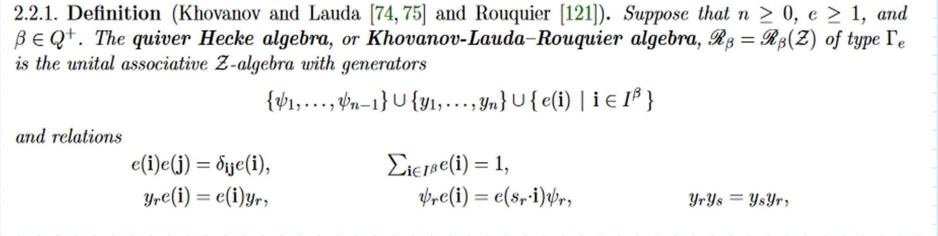
1.1.1. Definition (Hu-Mathas [57, Definition 2.2]). *Fix integers $n \geq 0$ and $\ell \geq 1$. The **cyclotomic Hecke algebra of type A**, with **Hecke parameter** $v \in \mathcal{Z}^\times$ and **cyclotomic parameters** $Q_1, \ldots, Q_\ell \in \mathcal{Z}$, is the unital associative $\mathcal{Z}$-algebra $\mathscr{H}_n = \mathscr{H}_n(\mathcal{Z}, v, Q_1, \ldots, Q_\ell)$ with generators $L_1, \ldots, L_n$, $T_1, \ldots, T_{n-1}$ and relations*

$$\prod_{l=1}^{\ell}(L_1 - Q_l) = 0, \qquad (T_r + v^{-1})(T_r - v) = 0, \qquad L_{r+1} = T_r L_r T_r + T_r,$$
$$L_r L_t = L_t L_r, \qquad T_r T_s = T_s T_r \text{ if } |r - s| > 1,$$
$$T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1}, \qquad T_r L_t = L_t T_r \text{ if } t \neq r, r+1,$$

where $1 \leq r < n$, $1 \leq s < n-1$ and $1 \leq t \leq n$.

$H_n^{\hat{\Lambda}}(\mathbb{F}, v) = H_n(\mathbb{F}, v, v^{k_1}, \ldots, v^{k_e})$

$(k_1, \ldots, k_e) \in I_e^{\ell}$ a choice of multicharge for Λ

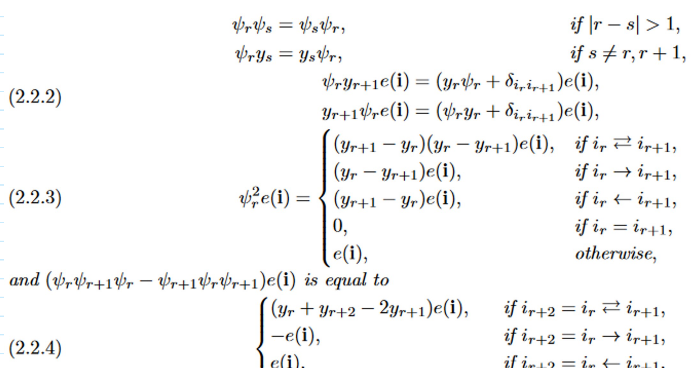


2.2.1. Definition (Khovanov and Lauda [74, 75] and Rouquier [121]). *Suppose that $n \geq 0$, $e \geq 1$, and $\beta \in Q^+$. The **quiver Hecke algebra**, or **Khovanov–Lauda–Rouquier algebra**, $\mathscr{R}_\beta = \mathscr{R}_\beta(\mathcal{Z})$ of type Γ_e is the unital associative $\mathcal{Z}$-algebra with generators*

$$\{\psi_1, \ldots, \psi_{n-1}\} \cup \{y_1, \ldots, y_n\} \cup \{\, e(\mathbf{i}) \mid \mathbf{i} \in I^\beta \,\}$$

and relations

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}} e(\mathbf{i}), \qquad \textstyle\sum_{\mathbf{i} \in I^\beta} e(\mathbf{i}) = 1,$$
$$y_r e(\mathbf{i}) = e(\mathbf{i}) y_r, \qquad \psi_r e(\mathbf{i}) = e(s_r \cdot \mathbf{i}) \psi_r, \qquad y_r y_s = y_s y_r,$$



$$\text{(2.2.2)} \quad \begin{aligned} \psi_r \psi_s &= \psi_s \psi_r, && \text{if } |r - s| > 1,\\ \psi_r y_s &= y_s \psi_r, && \text{if } s \neq r, r+1,\\ \psi_r y_{r+1} e(\mathbf{i}) &= (y_r \psi_r + \delta_{i_r i_{r+1}}) e(\mathbf{i}), &&\\ y_{r+1} \psi_r e(\mathbf{i}) &= (\psi_r y_r + \delta_{i_r i_{r+1}}) e(\mathbf{i}), && \end{aligned}$$

$$\text{(2.2.3)} \quad \psi_r^2 e(\mathbf{i}) = \begin{cases} (y_{r+1} - y_r)(y_r - y_{r+1}) e(\mathbf{i}), & \text{if } i_r \rightleftarrows i_{r+1},\\ (y_r - y_{r+1}) e(\mathbf{i}), & \text{if } i_r \to i_{r+1},\\ (y_{r+1} - y_r) e(\mathbf{i}), & \text{if } i_r \leftarrow i_{r+1},\\ 0, & \text{if } i_r = i_{r+1},\\ e(\mathbf{i}), & \text{otherwise}, \end{cases}$$

and $(\psi_r \psi_{r+1} \psi_r - \psi_{r+1} \psi_r \psi_{r+1}) e(\mathbf{i})$ *is equal to*

$$\text{(2.2.4)} \quad \begin{cases} (y_r + y_{r+2} - 2y_{r+1}) e(\mathbf{i}), & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1},\\ -e(\mathbf{i}), & \text{if } i_{r+2} = i_r \to i_{r+1},\\ e(\mathbf{i}), & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}, \end{cases}$$

$$R_n^{\Lambda}(\Gamma_e, \mathbb{F}) = \bigoplus_{ht(\beta) = n} \frac{R_\beta(\mathbb{F})}{\langle y_1^{\langle \Lambda, \alpha_{i_1} \rangle} e(\vec{i}) \mid \vec{i} \in Seq(\beta) \rangle}$$

Thrm 1 (Graded Isomorphism): $\exists$ alg iso

$$\Phi : R_n^{\Lambda}(\Gamma_e, \mathbb{F}) \xrightarrow{\sim} H_n^{\Lambda}(\mathbb{F}, v) \big|_{qchar(v) = e}$$

"Pf": - Recall in Pavel's talk in s.s case

$$H_n^{\Lambda} = \bigoplus_{\vec{\lambda} \in Std(\mathcal{P}_n^{\Lambda})} H_{\vec{\lambda}}, \quad H_{\vec{\lambda}} = \{h \mid L_r h = v^{c_r^{\mathbb{Z}}(\vec{\lambda})} h\}$$

- In general, only have gen eigenspace decomp

$$H_n^{\Lambda} = \bigoplus_{\vec{i} \in I_e^n} \widetilde{H}_{\vec{i}}, \quad \widetilde{H}_{\vec{i}} = \{h \mid (L_r - v^{i_r})^m h = 0\}$$

$\rightsquigarrow$ gives idempotents $F_{\vec{i}}$ in H_n^{Λ}

- Explicitly, $F_{\vec{i}} = \sum_{Res(\vec{\lambda}) = \vec{i}} F_{\vec{\lambda}}$ ⇙ defined in Pavel's talk

$Res(\vec{\lambda}) = (c_1^{\mathbb{Z}}(\vec{\lambda}) \bmod e, \ldots, c_n^{\mathbb{Z}}(\vec{\lambda}) \bmod e)$

Rem: In Pavel's talk s.s $\longleftrightarrow$ content separated so each $\vec{\lambda}$ has unique $\vec{i}$, i.e. $F_{Res(\vec{\lambda})} = F_{\vec{\lambda}}$

and $(\psi_r\psi_{r+1}\psi_r - \psi_{r+1}\psi_r\psi_{r+1})e(\mathbf{i})$ *is equal to*

$$\begin{cases} (y_r + y_{r+2} - 2y_{r+1})e(\mathbf{i}), & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1}, \\ -e(\mathbf{i}), & \text{if } i_{r+2} = i_r \rightarrow i_{r+1}, \\ e(\mathbf{i}), & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.2.4}$$

Isomorphism Theorem 2

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Defining Φ: • $\Phi(e(\vec{i})) = F_{\vec{i}}$

- $\Phi(y_r e(\vec{i})) = v^{-i_r}(L_r - v^{i_r}) F_{\vec{i}}$
- $\Phi(\psi_s e(\vec{i})) = (T_s + P_s(\vec{i})) \frac{1}{Q_s(\vec{i})} F_{\vec{i}}$

$P_s(\vec{i}), Q_s(\vec{i})$ power series in $\Phi(y_r e(\vec{i}))$ and $\Phi(y_{r+1} e(\vec{i}))$, becomes poly b/c $\Phi(y_r e(\vec{i}))$ nilpotent

- BK then checked all relations hold by hand, similarly w/ inverse map
- Mathas reduces to s.s. case via a modular system

Cor 2: $\exists$ non-trivial grading on H_n^Λ

$|F_{\vec{i}}| = 0$, $|\Phi(y_r e(i))| = 2$, $|\Phi(\psi_s e(i))| = -C_{i_s, i_{s+1}}$

Cor 3: Let $v, v' \in \mathbb{F}$ s.t. $\mathrm{qchar}(v) = \mathrm{qchar}(v') = e$

$$H_n^\Lambda(\mathbb{F}, v) \cong H_n^\Lambda(\mathbb{F}, v')$$

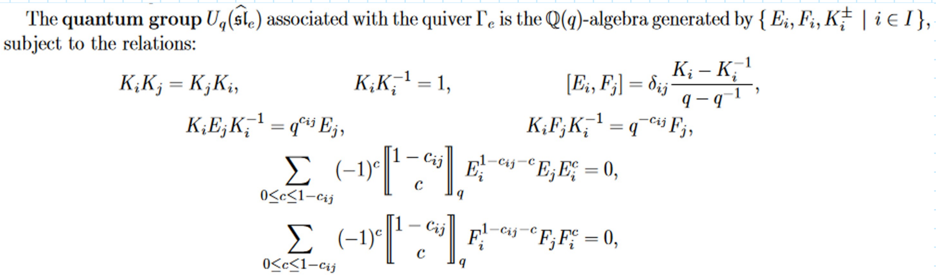
Rem: If $\mathbb{F} = \overline{\mathbb{F}_p}$, $v = 1$, $\mathrm{qchar}(v) = p$

If $\mathbb{F} = \mathbb{C}$, $v' = e^{2\pi i/p}$, $\mathrm{qchar}(v') = p$. So

$$H_n^\Lambda(\overline{\mathbb{F}_p}, 1) \overset{?}{\cong} H_n^\Lambda(\mathbb{C}, e^{2\pi i/p})$$

No! Φ depends on $\mathbb{F}$! But very close

2. $U_q(\hat{\mathfrak{sl}}_e)$ and it's Fock space



The **quantum group** $U_q(\widehat{\mathfrak{sl}}_e)$ associated with the quiver Γ_e is the $\mathbb{Q}(q)$-algebra generated by $\{ E_i, F_i, K_i^{\pm} \mid i \in I \}$, subject to the relations:

$$K_i K_j = K_j K_i, \qquad K_i K_i^{-1} = 1, \qquad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$K_i E_j K_i^{-1} = q^{c_{ij}} E_j, \qquad K_i F_j K_i^{-1} = q^{-c_{ij}} F_j,$$

$$\sum_{0 \le c \le 1 - c_{ij}} (-1)^c \begin{bmatrix} 1 - c_{ij} \\ c \end{bmatrix}_q E_i^{1-c_{ij}-c} E_j E_i^c = 0,$$

$$\sum_{0 \le c \le 1 - c_{ij}} (-1)^c \begin{bmatrix} 1 - c_{ij} \\ c \end{bmatrix}_q F_i^{1-c_{ij}-c} F_j F_i^c = 0,$$

Def: Λ-Fock space $\mathcal{F}_{\mathcal{Z}}^\Lambda$ is the free $\mathcal{Z}$-mod w/ basis $\{ |\vec{\lambda}\rangle \mid |\vec{\lambda}\rangle \in P^\Lambda = \bigcup_{n \geq 0} P_n^\Lambda \}$

Def: For $\vec{\mu} \in P_n^\Lambda$, a node A is an addable node of λ if $\vec{\mu} \cup \{A\} \in P_{n+1}^\Lambda$. Similarly w/ removable.

Fock Space

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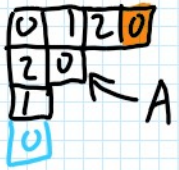
Def: node A is n i-node if cont(A) mod e = i

If A is an add/removable i-node of $\vec{\mu} \in P_n^\Lambda$, let

- $d_R(\vec{\mu}, A) = |\{B \in Add_i(\vec{\mu}) \mid B > A\}| - |\{B \in Rem_i(\vec{\mu}) \mid B > A\}|$
- $d^L(\vec{\mu}, A) = |\{C \in Add_i(\vec{\mu}) \mid C < A\}| - |\{C \in Rem_i(\vec{\mu}) \mid C < A\}|$

is below A in partition corr to A or in another partition to the right of partition corr to A

- $d_i(\vec{\mu}) = |Add_i(\vec{\mu})| - |Rem_i(\vec{\mu})|$

Ex: $\vec{\mu} =$ 

$\Lambda = \Lambda_0$

$k_1 = 0$

$e = 3$ $d_0(\vec{\mu}) = -1$

$\Rightarrow d^R(A) = 1 - 0$

$d^L(A) = 0 - 1$

Thrm 4 (Hayashi): Suppose $\Lambda \in P^+$. Then $\mathcal{F}^\Lambda_{\mathbb{Q}(q)}$ is an integrable $U_q(\hat{sl}_e)$ module where

- $E_i|\vec{\lambda}\rangle = \sum_{A \in Rem_i(\vec{\lambda})} q^{d_R(\vec{\lambda}, A)} |\vec{\lambda} - A\rangle$
- $F_i|\vec{\lambda}\rangle = \sum_{A \in Add_i(\vec{\lambda})} q^{-d^L(\vec{\lambda}, A)} |\vec{\lambda} + A\rangle$
- $K_i|\vec{\lambda}\rangle = q^{d_i(\vec{\lambda})} |\vec{\lambda}\rangle$

Ex: $\vec{\lambda} =$ 

$d_0(\vec{\lambda}) = 0$

$F_0(\vec{\lambda}) = q|(2,2,1,1)\rangle$

$E_0(\vec{\lambda}) = q|(2,1,1)\rangle$

- $E_0 F_0|\vec{\lambda}\rangle = q^{-1}(q|(2,1,1,1)\rangle) + q|(2,2,1)\rangle$
- $F_0 E_0|\vec{\lambda}\rangle = q|(2,2,1)\rangle + q^{-1}(q|(2,1,1,1)\rangle))$

$\Rightarrow [E_0, F_0]|\vec{\lambda}\rangle = 0$

- $K_0|\vec{\lambda}\rangle = q^0|\vec{\lambda}\rangle$

$\Rightarrow \frac{K_0 - K_0^{-1}}{q - q^{-1}}|\vec{\lambda}\rangle = 0$

Rem: Write $\Lambda = \Lambda_{1a} + \dots + \Lambda_{k\ell}$. Then as ↑ $U_q(\hat{sl}_e)$-mod

$$\mathcal{F}^\Lambda_{\mathbb{Q}(q)} \cong \mathcal{F}^{\Lambda_{k_1}}_{\mathbb{Q}(q)} \otimes \dots \otimes \mathcal{F}^{\Lambda_{k\ell}}_{\mathbb{Q}(q)}$$

L(\Lambda)

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Def $v \in \mathcal{F}^{\Lambda}_{\mathbb{Q}(q)}$ has weight $wt(v) = \alpha$ if

$$K_i v = q^{(\alpha,\alpha_i)} v \quad \forall i \in I_e$$

-Note for $|\phi_\ell\rangle = (\phi| \dots |\phi) \in P_n^{\Lambda}$ of level ℓ

$$K_i |\phi_\ell\rangle = q^{d_i(|\phi_\ell\rangle)} |\phi_\ell\rangle = q^{\#\Lambda_i \text{ appears in } \Lambda} |\phi_\ell\rangle$$

$$(\boxed{k_i}|\dots|) \quad = q^{(\Lambda,\alpha_i)} |\phi_\ell\rangle \Rightarrow |\phi_\ell\rangle \text{ has wt } \Lambda$$

- Clear that $E_i |\phi_\ell\rangle = 0 \ \forall i$ as nothing to remove

Def $L(\Lambda) = U_q(\widehat{sl_e}) |\phi_\ell\rangle$

Lemma 5 (1) $L(\Lambda)$ is h.w of weight Λ

(2) $L(\Lambda)$ is integrable $\Rightarrow$ simple

(3) $L(\Lambda)$ is the unique int $U_q(\widehat{sl_e})$-mod of wt Λ

Pf: (1) ✓ (2) Int $= E_i^n v = F_i^n v = 0 \quad \forall v, m, n >> 0$

E_i = remove nodes ✓ F_i = add i-nodes, but $|\vec{\lambda} + A\rangle$ for i-node A has fewer addable i-nodes ✓

(3) Find any book on crystal basis

3. Cat of $L(\Lambda)$

-Recall we had Ind, Res functors from

$$R(\alpha) \otimes R(\beta) \hookrightarrow R(\alpha + \beta)$$

-Now let i-Ind, i-Res be Ind, Res from

$$R^{\Lambda}(\alpha) \otimes R^{\Lambda}((i)) \hookrightarrow R^{\Lambda}((\alpha, i))$$

Lemma 6: (1) i-Ind, i-Res are exact

(2) i-Ind is biadjoint to i-Res

-Let $[\mathrm{Proj}_n^{\Lambda}(e)] = K_{\oplus}(R_n^{\Lambda}(\mathbb{F}_e, \mathbb{F}) \text{-gmod}) \otimes_{\mathbb{Z}} \mathbb{Q}(q)$

$[\mathrm{Rep}_n^{\Lambda}(e)] = K_0(R_n^{\Lambda}(\mathbb{F}_e, \mathbb{F}) \text{- gmod}) \otimes_{\mathbb{Z}} \mathbb{Q}(q)$

Thrm 7 (Cat Theorem): Let $\Lambda \in P^+$. Then letting $E_i = i\text{-Res}$, $F_i = q \circ i\text{-Ind} \circ K_i^{-1}$, we have

$$\bigoplus_{n \geq 0} [\mathrm{Proj}_n^{\Lambda}(e)] \cong L(\Lambda) \cong \bigoplus_{n \geq 0} [\mathrm{Rep}_n^{\Lambda}(e)]$$

as $U_q(\widehat{sl_e})$-mod

Categorification of L(\Lambda)

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Pf: Let $K_n^\Lambda = \{\vec{\lambda} \in P_n^\Lambda \mid D^{\vec{\lambda}} \neq 0\}$ ← simples. Then $[\mathrm{Proj}^\Lambda(e)]$ has basis $\{[P^{\vec{\mu}}] \mid \vec{\mu} \in K_n^\Lambda\}$. Consider

$$\begin{tikzcd}
{[\mathrm{Proj}_n^\Lambda(e)]} \arrow[r, "e_q"] & \mathcal{F}_n^\Lambda = \mathbb{Q}(q) P_n^\Lambda \arrow[d, "d_q"] \\
 & {[\mathrm{Rep}_n^\Lambda(e)]}
\end{tikzcd}$$

- $e_q([P^{\vec{\mu}}]) = \sum_{\vec{\lambda} \in P_n^\Lambda} [P^{\vec{\mu}} : S^{\vec{\lambda}}] |\vec{\lambda}\rangle$
- $d_q(|\vec{\lambda}\rangle) = [S^{\vec{\lambda}}]$

Step 1: d_q is a $U_q(\hat{sl}_e)$-morph

Prop (BK): $[i\text{-Res}\, S^{\vec{\lambda}}] = \sum_{A \in \mathrm{Rem}_i(\vec{\lambda})} q^{d_A(\vec{\lambda})} [S^{\vec{\lambda}-B}]$

$$[i\text{-Ind}\, S^\lambda (1-d_i; |\vec{\lambda}\rangle))] = \sum_{A \in \mathrm{Add}_i(\vec{\lambda})} q^{-d_A(\vec{\lambda})} [S^{\vec{\lambda}+A}]$$

Step 2: e_q is a $U_q(\hat{sl}_e)$-morph

- Notice $d_q(|\vec{\lambda}\rangle) = \sum_{\vec{\mu} \in K_n^\Lambda} [S^{\vec{\lambda}} : D^{\vec{\mu}}] [D^{\vec{\mu}}]$

$$= \sum_{\vec{\mu} \in K_n^\Lambda} d_{\vec{\lambda}\vec{\mu}}(q) [D^{\vec{\mu}}].$$ By BH-reciproc

$$[P^{\vec{\mu}} : S^{\vec{\lambda}}] = [S^{\vec{\lambda}} : D^{\vec{\mu}}] = d_{\vec{\lambda}\vec{\mu}}$$

$$\Rightarrow e_q = d_q^T = d_q^* : [\mathrm{Rep}_n^\Lambda(e)] \longrightarrow \mathcal{F}_n^{\Lambda *}$$

after $[\mathrm{Rep}_n^\Lambda(e)] \xrightarrow{\sim} [\mathrm{Proj}_n^\Lambda(e)]$ via Cartan

$\mathcal{F}_n^{\Lambda *} \xrightarrow{\sim} \mathcal{F}_n^\Lambda$, via dual $|\lambda\rangle^* \longmapsto |\lambda\rangle$

- Check $\langle F_i |\lambda\rangle^*, |\mu\rangle\rangle_{\mathrm{dual}} \overset{(?)}{=} \langle |\lambda\rangle^*, E_i |\mu\rangle\rangle_{\mathrm{dual}}$

LHS: $\langle \sum_{A \in \mathrm{Add}_i(\lambda)} |\lambda + A\rangle^*, |\mu\rangle\rangle_d = \begin{cases} 1 & \text{if } \lambda + A = \mu \\ 0 & \text{otherwise} \end{cases}$

$q=1$

RHS: $\langle |\lambda\rangle^*, \sum_{B \in \mathrm{Rem}_i(\mu)} |\vec{\mu} - B\rangle\rangle = \begin{cases} 1 & \text{if } \lambda = \mu - B \\ 0 & \text{otherwise} \end{cases}$

$q=1$

$$\langle e_q(E_i \cdot Y), X\rangle_{\mathrm{dual}} \overset{\mathrm{dual}}{=} \langle E_i \cdot Y, d_q(X)\rangle_C$$

$$\overset{\mathrm{def}}{=} \langle i\text{-Res}(Y), X\rangle_C \overset{\text{bi-adj}}{=\!=} \langle Y, i\text{-Ind}\, d_q(X)\rangle_C$$

$$\overset{\text{Step 1}}{=} \langle Y, d_q(i\text{-Ind} X)\rangle_C \overset{\mathrm{dual}}{=} \langle d_q^* Y, F_i X\rangle_{\mathrm{dual}}$$

$$= \langle E_i \cdot e_q(Y), X\rangle_{\mathrm{dual}}$$

$$= \langle E_i \cdot e_q(Y), x \rangle_{dual}$$

Categorification of L(\Lambda)

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Step 3: e_q is an iso

- Note that $d_q(|\vec{\lambda}\rangle) = \sum_{\vec{\mu} \in K_n^\wedge} [S^{\vec{\lambda}} : D^{\vec{\mu}}][D^{\vec{\mu}}]$ written in basis $\{|\lambda\rangle\}_{\lambda \in P_n^\wedge}$ of $\mathcal{F}_n^\wedge$ and basis $\{[D^\mu]\}_{\mu \in K_n^\wedge}$ is literally the matrix D^T where $D = (d_{\lambda\mu})_{\substack{\lambda \in P_n^\wedge \\ \mu \in K_n^\wedge}} \Rightarrow e_q = {d_q}^T = D$
- From Dinushi's talk D has 1's on diagonal $\Rightarrow$ full rank $\Rightarrow e_q$ is inj
- Because $e_q(p_{\emptyset e}) = |\phi_e\rangle$ and e_q is a $U_q(\widehat{sl_e})$-morph + $[\mathrm{Proj}^\wedge(e)]$ is cyclic $U_q(\widehat{sl_e})$-mod??
  $\Rightarrow \operatorname{im} e_q \subseteq L(\Lambda)$.
  $L(\Lambda)$ simple $\Rightarrow \operatorname{im} e_q = L(\Lambda)$. Dualize to get corr statement for $[\mathrm{Rep}_n^\wedge(e)]$

Thrm 8 (Ariki): When char $\mathbb{F} = 0$, $q = 1$, the iso in Cat Theorem sends the basis $\{[\underline{P^{\vec{\mu}}}]\}$ of indecomp projective $\bigoplus_{n \geq 0} H_n^\wedge(\mathbb{F})$-modules to canonical basis of $L_1(\Lambda)$ of $U(\widehat{sl_e})$.

Thrm 9 (BK): When char $\mathbb{F} = 0$, the iso in Cat Theorem sends the basis $\{q^{-\mathrm{def}\,\vec{\mu}}[P^{\vec{\mu}}] \mid \vec{\mu} \in K^\wedge\}$ of indecomp self-dual projective $\bigoplus_{n \geq 0} H_n^\wedge(\mathbb{F})$-gmod to canonical basis of $L(\Lambda)$ of $U_q(\widehat{sl_e})$.

Rem: There are efficient algor to compute canonical basis of $L(\Lambda)$ such as LLT

Cor 10: $\exists$ explicit combinatorial description for $K_n^\wedge = \{\vec{\mu} \mid D^{\vec{\mu}} \neq 0\}$

Pf: $[P^{\vec{\mu}}] \hookrightarrow$ canonical basis = (lower) crystal basis and crystal graph of $L(\Lambda)$ is well-known/studied.

Decomposition Multiplicities

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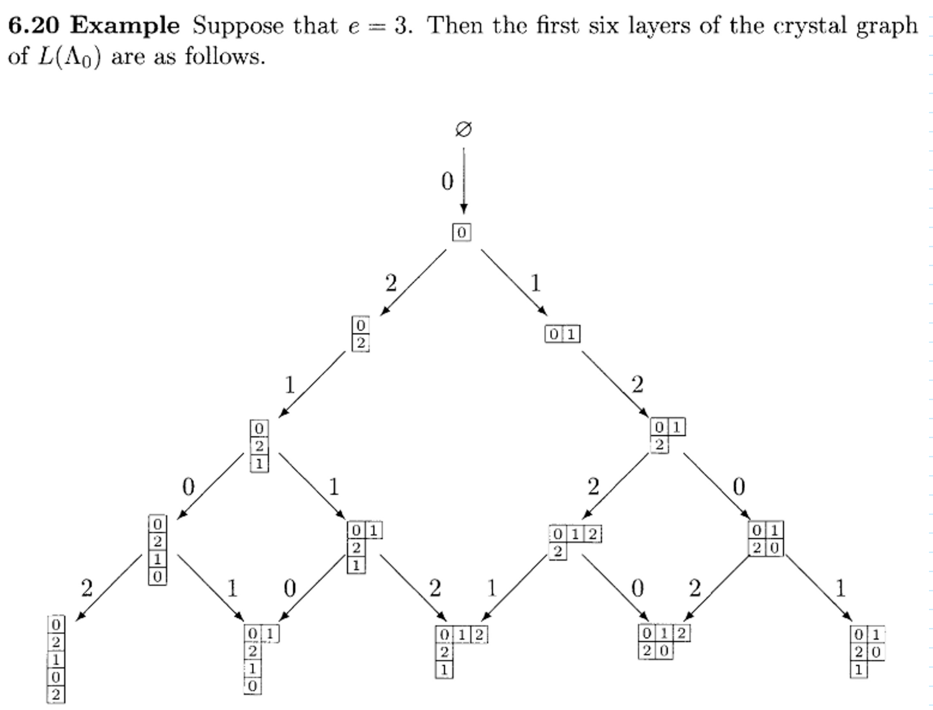

6.20 Example Suppose that $e = 3$. Then the first six layers of the crystal graph of $L(\Lambda_0)$ are as follows.

Cor 11: Let $\{b_{\vec{\lambda}}\}$ be the canonical basis for $L(\Lambda)$. Write $b_{\vec{\lambda}} = |\vec{\lambda}\rangle + \sum_{\substack{\vec{\mu} \in K_n \\ \vec{\mu} < \vec{\lambda}}} b_{\vec{\mu}\vec{\lambda}}(q)|\vec{\mu}\rangle$

Then $[S^{\vec{\mu}} : D^{\vec{\lambda}}]_q = b_{\vec{\mu}\vec{\lambda}}(q)$.

Pf: Recall if $C = ([P^{\vec{\mu}} : S^{\vec{\lambda}}])$, then $C = D^t D$. $D = ([S^{\vec{\lambda}} : D^{\vec{\mu}}])$. By Cat Thm

$$\begin{tikzcd}
{[\mathrm{Proj}^{\Lambda}(e)]} \arrow[r, "e_q = D", "\sim"'] \arrow[dr, "\sim", "{c_q = C \; (= \text{natural inclusion})}"'] & L(\Lambda) \arrow[d, "d_q = D^t", "\wr"'] \\
 & {[\mathrm{Rep}^{\Lambda}(e)]}
\end{tikzcd}$$

By graded BH reciprocity

$$[S^{\vec{\mu}} : D^{\vec{\lambda}}]_q = [P^{\vec{\lambda}} : S^{\vec{\mu}}]_q$$

$$= [e_q^{-1} c_q^{-1}(P^{\vec{\lambda}}) : d_q^{-1}(S^{\vec{\mu}})]_q$$

$$= [e_q^{-1}(P^{\vec{\lambda}}) : |\mu\rangle]_q = [b_{\vec{\lambda}} : |\mu\rangle]_q$$

$$= b_{\vec{\mu}\vec{\lambda}}(q)$$

Addendum

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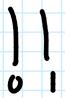
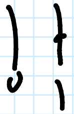
- $\mathbb{F}[S_n]$ is obtained from $H_n^\wedge(\mathbb{F}, v)$ by setting $v=1$, $\Lambda = \Lambda_0 \Rightarrow L_1 = [k_1]_v = 0$ (revert back to [m] def of Q_i)
- Suppose char $\mathbb{F} = 2$, $n=2$, then

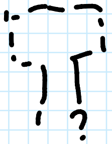
$$\overline{\mathbb{F}}_2[S_2] \cong R_2^{\Lambda_0}(\hat{A}_1, \overline{\mathbb{F}}_2) = \overline{\mathbb{F}}_2 \cdot$$



$\oplus\ \overline{\mathbb{F}}_2$

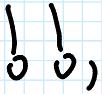


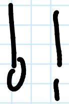
as



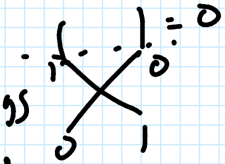
$= 0$ since $\langle \Lambda_0, \alpha_1 \rangle = 0$

$\Rightarrow$

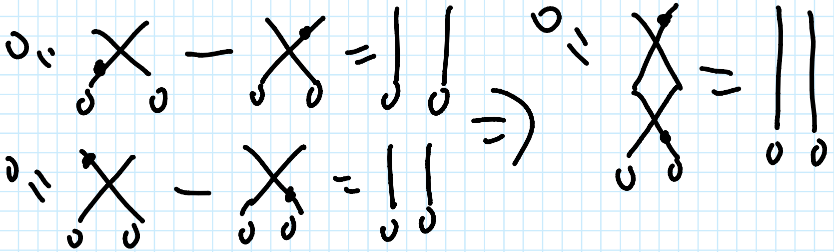




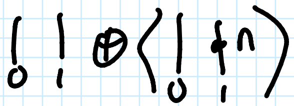
+dots span as crossings



- $\langle \Lambda_0, \alpha_0 \rangle = 1 \Rightarrow$ no dots on first 0 strand
- NilHecke relation

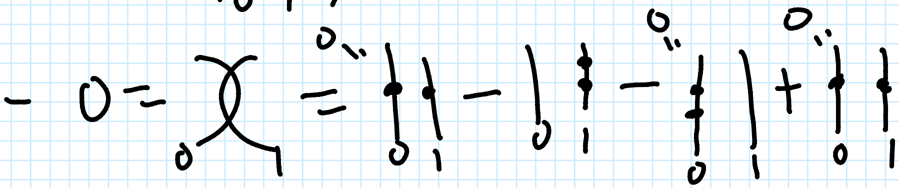


$\Rightarrow$



span

-



$\Rightarrow\ \overline{\mathbb{F}}_2(S_2) \cong \overline{\mathbb{F}}_2[z]/(z^2)$

Coxeter presentation

$\overline{\mathbb{F}}_2[S_2] = \mathbb{k}[s_1]/(s_1^2 - 1)$, but $(s_1 - 1)^2 = s_1^2 - 1$